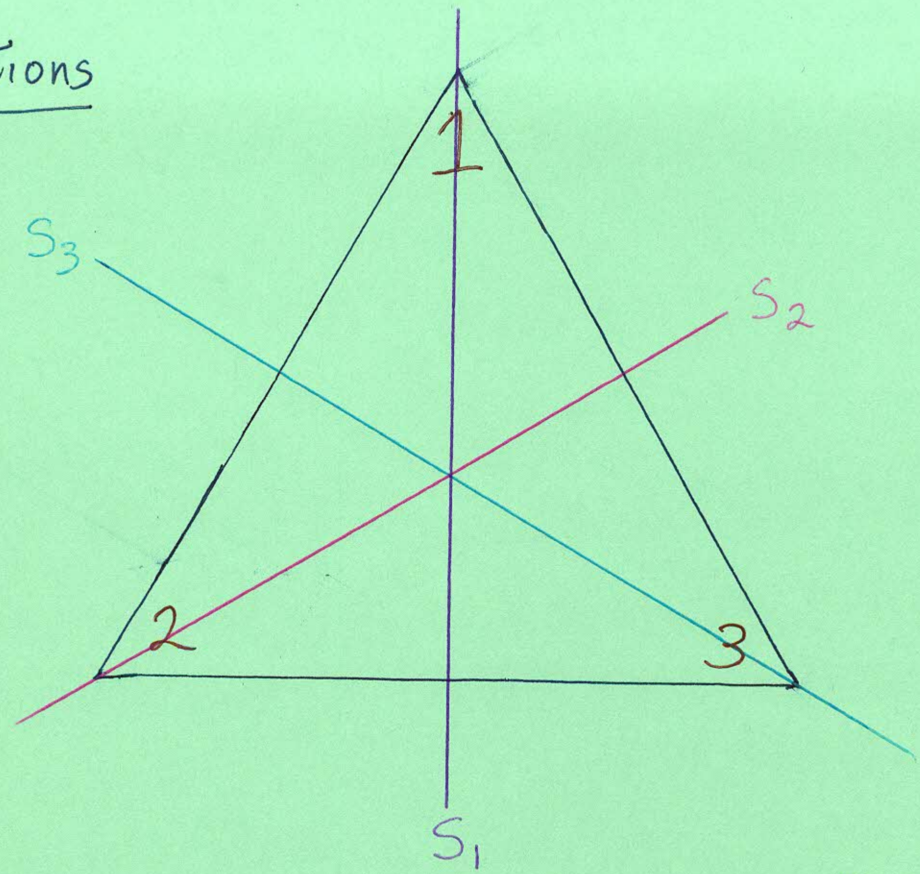
Reflections



There are three lines we can reflect across while maintaining the shape, let's call these reflections S_1, S_2, and S_3. In cycle notation:

$$S_1 = (1)(2\,3) = (2\,3)$$

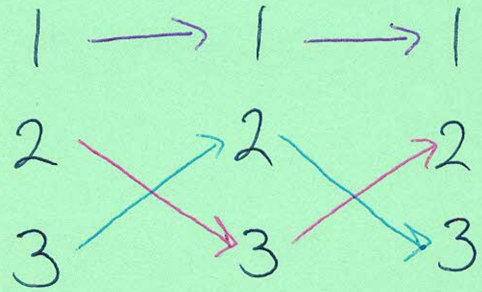
$$S_2 = (2)(1\,3) = (1\,3)$$

$$S_3 = (3)(1\,2) = (1\,2)$$

Of course, reflecting across a line, then across it again should bring us back to the starting point, i.e.,

$$(S_1)^2 = e,\ (S_2)^2 = e,\ \&\ (S_3)^2 = e$$

Let's verify this for S_1 w/ cycles:



$$(2\,3)\cdot(2\,3) = e$$

<u>Mixing the different operations.</u>

Since all of the rotations are based off of $R_{\frac{2\pi}{3}}$, let's just write $r = R_{\frac{2\pi}{3}}$ so that

$$r = R_{\frac{2\pi}{3}},\ r^2 = R_{\frac{4\pi}{3}} = R_{-\frac{2\pi}{3}} = r^{-1},\ R_{-\frac{4\pi}{3}} = \left(R_{-\frac{2\pi}{3}}\right)^2 = r^{-2}$$

$$= R_{\frac{2\pi}{3}} = r$$

<u>Let's start by mixing the S's:</u>

$$S_1 S_2 = (2\ 3)(1\ 3) = (1\ 3\ 2) = r^2$$

$$S_1 S_3 = (2\ 3)(1\ 2) = (1\ 2\ 3) = r$$

$$S_2 S_3 = (1\ 3)(1\ 2) = (1\ 3\ 2) = r^2$$

So, any time we pair 2 of the S's, we get some kind of rotation...

Notice that we get some behavior which you may not have seen before ... <u>noncommutativity</u>:

$$S_2 S_1 = (1\ 3)(2\ 3) = (1\ 2\ 3) = r \neq S_1 S_2$$

<u>Let's start mixing the r & S's now:</u>

$$r S_1 = (1\ 2\ 3)(2\ 3) = (1\ 3)(2) = (1\ 3) = S_2$$

$$r S_2 = (1\ 2\ 3)(1\ 3) = (1\ 2)(3) = (1\ 2) = S_3 = r^2 S_1$$

$$r S_3 = (1\ 2\ 3)(1\ 2) = (1)(2\ 3) = (2\ 3) = S_1$$

So, we can write S_2 & S_3 in terms of r & S_1:

$$S_2 = r S_1 \quad \& \quad S_3 = r^2 S_1$$

Thus, we can forget about S_2 & S_3 by using r & S_1. On that note, let's just write $s = S_1$.

Thus, we have found really 6 distinct motions:

$e = e$	$s = (2\ 3)$
$r = (1\ 2\ 3)$	$rs = (1\ 3)$
$r^2 = (1\ 3\ 2)$	$r^2 s = (1\ 2)$

To finish understanding how r & s work together, we need to try and switch their order.

$$sr = (2\ 3)(1\ 2\ 3) = (1\ 2) = r^2 s = r^{-1} s$$

likewise

$$sr^2 = (2\ 3)(1\ 3\ 2) = (1\ 3) = rs$$

$$sr^2 = sr^{-1}$$

These two equations are really the same since:

$$(sr = r^{-1}s)\cdot r \Rightarrow sr^2 = r^{-1}(sr) = r^{-1}(r^{-1}s)$$

$$= r^{-2}s = rs$$

So, using $sr^{-1} = rs$, we can make anything not in the list $\{e, r, r^2, s, rs, r^2 s\}$ inside the list.

So, these really are the only 6 motions.

We write D_3 for the <u>group</u> of symmetries of the equilateral triangle. We can <u>generate</u> all 6 motions using $r^3 = e$, $s^2 = e$, and $rs = sr^{-1}$, so we also write $D_3 = \langle r, s \mid r^3 = s^2 = e, rs = sr^{-1} \rangle$

There's one more way we can be sure we've got everything: look at all the ways to number the triangle's corners:

The first corner has 3 choices
the second corner has 2 choices
the third corner has 1 choice

So, there are $3! = 6$ choices!

In general, there is a group called the <u>symmetric group on n letters</u> which consists of all ways to permute n things. Using $1, 2, \cdots, n$ for those things, we write them in the cycle notation, as we have been. We denote this full group by S_n.

In the case, $n=3$, we have $D_3 = S_3$.